

ON A LOWER BOUND FOR THE TIME CONSTANT OF FIRST-PASSAGE PERCOLATION

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Abstract

We consider the Bernoulli first-passage percolation on \mathbb{Z}^d ($d \geq 2$). That is, the edge passage time is taken independently to be 1 with probability $1 - p$ and 0 otherwise. Let $\mu(p)$ be the time constant. We prove in this paper that

$$\mu(p_1) - \mu(p_2) \geq \frac{\mu(p_2)}{1 - p_2}(p_2 - p_1)$$

for all $0 \leq p_1 < p_2 < 1$ by using Russo's formula.

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1 Introduction and statement of the results.

We begin with the general first-passage percolation on \mathbb{Z}^d . Let $\{t(e) : e \in \mathbb{Z}^d\}$ be a sequence of i.i.d. positive random variables with common distribution F , $t(e)$ is the random passage time of edge e and F is the edge-passage distribution of the model. For any path $\gamma = \{e_1, e_2, \dots, e_n\}$, the passage time of γ is

$$T(\gamma) := \sum_{k=1}^n t(e_k).$$

For any vertices $u, v \in \mathbb{Z}^d$ and vertex sets $A, B \subset \mathbb{Z}^d$, let

$$T(u, v) := \inf_{\gamma \ni u, v} T(\gamma); \quad T(A, B) := \inf_{u \in A, v \in B} T(u, v)$$

be the passage time from u to v and the passage time from A to B .

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Let 0 be the origin of \mathbb{Z}^d , $\hat{e}_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$ and $H_n = \{u = (u_1, u_2, \dots, u_n) \in \mathbb{Z}^d : u_1 = n\}$. Define

$$a_{0,n} := T(0, n\hat{e}_1), \quad b_{0,n} := T(0, H_n).$$

To restrict $a_{0,n}, b_{0,n}$ on cylinders, let

$$\Gamma^{cyl}(0, n\hat{e}_1) = \{\gamma : 0, n\hat{e}_1 \in \gamma \text{ and } \forall u \in \gamma, 0 \leq u_1 \leq n\}$$

$$\Gamma^{cyl}(0, H_n) = \{\gamma : 0 \in \gamma, \gamma \cap H_n \neq \emptyset, \text{ and } \forall u \in \gamma, 0 \leq u_1 \leq n\}$$

and define

$$t_{0,n} := \inf_{\gamma \in \Gamma^{cyl}(0, n\hat{e}_1)} T(\gamma); \quad s_{0,n} := \inf_{\gamma \in \Gamma^{cyl}(0, H_n)} T(\gamma).$$

The time constant μ of the model is the common limit of $\theta_{0,n}/n$ when $n \rightarrow \infty$ for $\theta = a, b, t$ or s . Here we will not introduce all the detailed situations for the above convergence under various moment conditions of F , and only point out that, in most cases, for $\theta = a, b, t$ or s ,

$$\frac{\theta_{0,n}}{n} \rightarrow \mu = \mu(F) \quad \text{a.s. as } n \rightarrow \infty. \quad (1.1)$$

For the details on the convergence to μ , one may refer to [4, 6, 7, 8].

It is straightforward that $\theta_{0,n}, \theta = a, b, t$ or s , depends on the states of infinitely many edges. The following is another limit representation of μ given by Grimmett and Kesten [3], from which, μ is represented as the limit of random variables which only depend on the states of finitely many edges.

For any fixed $n \geq 1$, let $B_n = \{u \in \mathbb{Z}^d : 0 \leq u_i \leq n, 1 \leq i \leq d\}$ be the box with side length n . Let

$$\phi_{0,n} = \inf\{T(\gamma) : \gamma \text{ is a path in } B_n \text{ from } \{0\} \times [0, n]^{d-1} \text{ to } \{n\} \times [0, n]^{d-1}\}$$

Grimmett and Kesten [3] proved that, if the time-passage distribution F satisfying:

$$\int (1 - F(x))^4 dx < \infty \text{ for } d = 2; \text{ or } \int x^2 dF(x) < \infty \text{ for } d \geq 3$$

then

$$\frac{\phi_{0,n}}{n} \rightarrow \mu \quad \text{a.s. and in } L^1, \quad \text{as } n \rightarrow \infty. \quad (1.2)$$

The first problem for time constant μ is: when will $\mu > 0$? Kesten [5] solved this problem for all $d \geq 2$ as:

$$\mu > 0 \Leftrightarrow F(0) < p_c(d), \quad (1.3)$$

where $F(0) = \mathbb{P}(t(e) = 0)$ and $p_c(d)$ be the critical probability for the general bond percolation on \mathbb{Z}^d .

Further study on μ is carried out to solve such a problem: How does $\mu = \mu(F)$ depend on the edge-passage distribution F ? Berg and Kesten [1] solved this problem in part. As our result is a further research in this direction, in the next paragraph, we introduce the results of Berg and Kesten in detail.

Let's begin with some notations. For any given edge-passage distributions F , let $\text{supp}(F) = \{x \geq 0 : F(x) > 0\}$ be the support of F , let $\lambda(F) = \inf \text{supp}(F)$. We say F is *useful*, if

$$\lambda(F) = 0 \text{ and } F(0) < p_c(d), \text{ or } \lambda(F) > 0 \text{ and } F(\lambda) < \vec{p}_c(d),$$

where $\vec{p}_c(d)$ is the critical probability for directed bond percolation on \mathbb{Z}^d . For two edge-passage distributions F and \tilde{F} , we say \tilde{F} is more *variable* than F , if

$$\int \varphi(x) d\tilde{F}(x) \leq \int \varphi(x) dF(x) \quad (1.4)$$

for all increasing convex function φ . Clearly, by the above definition, “ \tilde{F} is more *variable* than F ” is a weaker condition than “ \tilde{F} is *stochastically dominated* by F ”, note that the latter requires equation (1.4) hold for all increasing φ .

Theorem 1.1 [Berg and Kesten [1]]

(a) Let F and \tilde{F} be two edge-passage distribution functions, if \tilde{F} is more variable than F , then

$$\mu(\tilde{F}) \leq \mu(F);$$

(b) if, in addition, F is useful and $F \neq \tilde{F}$, then

$$\mu(\tilde{F}) < \mu(F).$$

Theorem 1.1 gives sufficient conditions for (strict) inequality between $\mu(\tilde{F})$ and $\mu(F)$, but for the difference $\mu(F) - \mu(\tilde{F})$, no information is provided. One may ask: what can we say for such a difference? In this paper, for the simplest case, i.e., under the following Bernoulli setting, we give a nontrivial lower bound for this difference.

From now on, we take $\{t(e) : e \in \mathbb{Z}^d\}$ to be the i.i.d. random variable sequence such that $t(e) = 1$ with probability $1 - p$ and $t(e) = 0$ with probability p , $p \in [0, 1]$. Write \mathbb{P}_p as the percolation measure and \mathbb{E}_p as its expectation. Write $\mu(p)$ as the corresponding time constant. By (1.3) and Theorem 1.1, $\mu(p)$ decreases strictly in p when $p \in [0, p_c(d))$, i.e.,

$$\frac{\mu(p_1)}{\mu(p_2)} > 1 \quad (1.5)$$

for all $0 \leq p_1 < p_2 < p_c(d)$.

Now, we state our main result as follows.

Theorem 1.2 *For the above Bernoulli first-passage percolation model, let $\mu(p)$ be its time constant. We have that $\mu(p)/(1-p)$ decreases in p and then*

$$\mu(p_1) - \mu(p_2) \geq \frac{\mu(p_2)}{1-p_2}(p_2 - p_1) \quad (1.6)$$

for all $0 \leq p_1 < p_2 \leq 1$.

Remark 1.1 *By the monotonicity of $\mu(p)/(1-p)$ and (1.3), when $0 \leq p_1 < p_2 < p_c(d)$, one has*

$$\frac{\mu(p_1)}{\mu(p_2)} \geq 1 + \frac{p_2 - p_1}{1 - p_2}. \quad (1.7)$$

This is a concretion of (1.5).

2 Proof of Theorem 1.2

To use the Russo's formula, we first give the definition of *pivotal* edges according to Grimmett [2]. For any edge e and configuration ω , let ω_e be the configuration such that $\omega_e(f) = \omega(f)$ for all $f \neq e$ and $\omega_e(e) = 1 - \omega(e)$.

Recall that $B_n = [0, n]^d \cap \mathbb{Z}^d$. Suppose that A be an event which only depends on edges of B_n . We say edge $e \in B_n$ is *pivotal* for pair (A, ω) , if

$$I_A(\omega) \neq I_A(\omega_e),$$

where I_A be the indicator function of A . Write $S_e(A)$ as the event that e is a pivotal edge for A , i.e.

$$S_e(A) = \{\omega : e \text{ is pivotal for pair } (A, \omega)\}. \quad (2.1)$$

By the above definition, $S_e(A)$ is independent of $t(e)$. Denote by $N(A)$ the number of pivotal edges of A , i.e.

$$N(A)(\omega) = |\{e \in B_n : \omega \in S_e(A)\}|. \quad (2.2)$$

Event A is called *increasing* if $\omega \in A$ and $\omega \leq \omega'$ imply $\omega' \in A$, where $\omega \leq \omega'$ means $\omega(e) \leq \omega'(e)$ for all e . The Russo's formula says that (in our setting), if A is increasing, then

$$\frac{d\mathbb{P}_p(A)}{dp} = -\mathbb{E}_p(N(A)). \quad (2.3)$$

Proof of Theorem 1.2: Firstly, by equation (1.2), we have

$$\mu(p) = \lim_{n \rightarrow \infty} \frac{E_p \phi_{0,n}}{n} \quad (2.4)$$

for all $p \in [0, 1]$.

For any integer $k \geq 1$, let $A_{n,k} = \{\phi_{0,n} \geq k\}$. Clearly, $A_{n,k}$ is increasing and only depends on edges in B_n . Rewrite $\mathbb{E}_p(\phi_{0,n})$ as

$$\mathbb{E}_p(\phi_{0,n}) = \sum_{k=1}^{\infty} \mathbb{P}_p(A_{n,k}). \quad (2.5)$$

For any $0 \leq p_1 < p_2 \leq 1$, by (2.4) and (2.5), we have

$$\begin{aligned} \mu(p_1) - \mu(p_2) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\infty} (\mathbb{P}_{p_1}(A_{n,k}) - \mathbb{P}_{p_2}(A_{n,k})) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\infty} \int_{p_1}^{p_2} -\frac{d\mathbb{P}_p(A_{n,k})}{dp} dp. \end{aligned} \quad (2.6)$$

Using the Russo's formula and the fact that $A_{n,k}$ is increasing, we have

$$\begin{aligned} \frac{d\mathbb{P}_p(A_{n,k})}{dp} &= -\mathbb{E}_p(N(A_{n,k})) = -\sum_{e \in B_n} \mathbb{P}_p(S_e(A_{n,k})) \\ &= -\frac{1}{1-p} \sum_{e \in B_n} \mathbb{P}_p(\{t(e) = 1\} \cap S_e(A_{n,k})) \\ &= -\frac{1}{1-p} \sum_{e \in B_n} \mathbb{P}_p(A_{n,k} \cap S_e(A_{n,k})) \\ &= -\frac{1}{1-p} \sum_{e \in B_n} \mathbb{P}_p(S_e(A_{n,k}) \mid A_{n,k}) \mathbb{P}_p(A_{n,k}) \\ &= -\frac{1}{1-p} \mathbb{E}_p(N(A_{n,k}) \mid A_{n,k}) \mathbb{P}_p(A_{n,k}). \end{aligned} \quad (2.7)$$

Note that the third equality comes from the independence of $t(e)$ and $S_e(A_{n,k})$.

To finish the proof, we have to give appropriate lower bound for $\mathbb{E}_p(N(A_{n,k}) \mid A_{n,k})$. To this end, for any configuration $\omega \in A_{n,k}$, we give lower bounds to $N(A_{n,k})(\omega)$ in the following two cases respectively: 1) $\phi_{0,n}(\omega) \geq k+1$; 2) $\phi_{0,n}(\omega) = k$.

We first deal with the case of $\phi_{0,n}(\omega) \geq k+1$. For any $e \in B_n$, because ω_e only differs from ω in edge e , the change from ω to ω_e can at most decrease $\phi_{0,n}$ by 1, this implies that $\phi_{0,n}(\omega_e) \geq k$, and $\omega_e \in A_{n,k}$. By the definition of pivotal edges, we know that e is not pivotal for $(A_{n,k}, \omega)$. So

$$N(A_{n,k})(\omega) = 0. \quad (2.8)$$

Now, we consider the case of $\phi_{0,n}(\omega) = k$. For any $e \in B_n$, if e is pivotal for $(A_{n,k}, \omega)$, we declare that $\omega(e) = 1$. Actually, if $\omega(e) = 0$, then the change from ω to ω_e will increase $\phi_{0,n}$, so we have $\phi_{0,n}(\omega_e) \geq \phi_{0,n}(\omega) = k$ and $\omega_e \in A_{n,k}$, this leads to a contradiction.

Suppose γ be a path in B_n from $\{0\} \times [0, n]^{d-1}$ to $\{n\} \times [0, n]^{d-1}$ with $T(\gamma)(\omega) = \phi_{0,n}(\omega) = k$. If $e \in \gamma$ satisfying $\omega(e) = 1$, then $T(\gamma)(\omega_e) = k-1$. This implies that $\phi_{0,n}(\omega_e) \leq k-1$ and $\omega_e \notin A_{n,k}$. Thus, by the definition of pivotal edges, e is pivotal for pair $(A_{n,k}, \omega)$.

By the arguments in the last two paragraphs, we have

$$N(A_{n,k})(\omega) \geq |\{e \in \gamma : \omega(e) = 1\}| = T(\gamma)(\omega) = k \quad (2.9)$$

for all $\omega \in A_{n,k}$.

Combining (2.8) and (2.9), we have

$$\begin{aligned} \mathbb{E}_p(N(A_{n,k}) \mid A_{n,k}) \mathbb{P}_p(A_{n,k}) &= \sum_{\omega \in A_{n,k}} N(A_{n,k})(\omega) \cdot \frac{\mathbb{P}_p(\omega)}{\mathbb{P}_p(A_{n,k})} \mathbb{P}_p(A_{n,k}) \\ &\geq \sum_{\{\omega : \phi_{0,n}(\omega) = k\}} k \cdot \mathbb{P}_p(\omega) \\ &= k \cdot \mathbb{P}_p(\{\omega : \phi_{0,n}(\omega) = k\}). \end{aligned} \quad (2.10)$$

Finally, by (2.6), (2.7) and (2.10), using the Fubini's theorem and the Fatou's lemma, we have

$$\begin{aligned} \mu(p_1) - \mu(p_2) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\infty} \int_{p_1}^{p_2} \frac{1}{1-p} \mathbb{E}_p(N(A_{n,k}) \mid A_{n,k}) \mathbb{P}_p(A_{n,k}) dp \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\infty} \int_{p_1}^{p_2} \frac{1}{1-p} k \cdot \mathbb{P}_p(\{\omega : \phi_{0,n}(\omega) = k\}) dp \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \int_{p_1}^{p_2} \frac{1}{1-p} \sum_{k=1}^{\infty} k \cdot \mathbb{P}_p(\{\omega : \phi_{0,n}(\omega) = k\}) dp \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \int_{p_1}^{p_2} \frac{1}{1-p} \mathbb{E}_p(\phi_{0,n}) dp \\ &\geq \int_{p_1}^{p_2} \frac{1}{1-p} \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_p(\phi_{0,n}) dp \\ &= \int_{p_1}^{p_2} \frac{\mu(p)}{1-p} dp \end{aligned} \quad (2.11)$$

for all $0 \leq p_1 < p_2 < 1$. Clearly, the inequality (2.11) is equivalent to the following differential inequality

$$\frac{d[\mu(p)/(1-p)]}{dp} \leq 0, \quad 0 \leq p < 1. \quad (2.12)$$

This gives that

$$\int_{p_1}^{p_2} \frac{\mu(p)}{1-p} dp \geq \frac{\mu(p_2)}{1-p_2} (p_2 - p_1)$$

for all $0 \leq p_1 < p_2 < 1$ and we finish the proof of Theorem 1.2. \square

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